Rational Approximation of Noisy Frequency Responses

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Abstract. Vector Fitting is a robust technique for transfer function synthesis in the frequency domain. This method combines the use of a Sanathanan-Koerner iteration and a well-chosen partial fraction basis. It is known that its convergence properties become impaired if the data is contaminated with noise, which results in poor fitting models. In this paper, an alternative approach is presented, and its performance is compared to the classical formulation.

1. Introduction

Rational function identification from measured or simulated data becomes increasingly important for the modeling of linear systems and devices. Nowadays, the Vector Fitting (VF) method [1] has become a standard approach in the field to calculate such a transfer function in a reliable way. In the VF method, the numerator $N(s)$ and denominator $D(s)$ of the transfer function are represented as a linear combination of P partial fractions, based on a prescribed set of poles $-a_p$, such that

$$R(s) = \frac{N(s)}{D(s)} = \frac{\sum_{p=1}^{P} c_p/(s+a_p)}{\tilde{c}_0 + \sum_{p=1}^{P} \tilde{c}_p/(s+a_p)} \qquad s = j2\pi f. \tag{1}$$

The denominator has an additional basis function which equals the constant value 1, and the coefficients c_p and $\tilde{c}_p$ represent the model coefficients. Given a set of Laplace data samples $(s_k, H(s_k))$, the transfer function should match the data in a least-squares (LS) sense, such that $R(s_k) \simeq H(s_k)$, for $k = 0, ..., K$. Some further improvements in terms of conditioning can be made by using a set of orthonormal rational functions, leading to the Orthonormal Vector Fitting (OVF) method [2].

The numerator and denominator of (1) can be factorized as follows

$$N(s) = \sum_{p=1}^{P} \frac{c_p}{s+a_p} = \frac{\prod_{p=1}^{P-1}(s+z_{p,n})}{\prod_{p=1}^{P}(s+a_p)} \tag{2}$$

$$D(s) = \tilde{c}_0 + \sum_{p=1}^{P} \frac{\tilde{c}_p}{s+a_p} = \frac{\prod_{p=1}^{P}(s+z_{p,d})}{\prod_{p=1}^{P}(s+a_p)} \tag{3}$$

and the transfer function $R(s)$ is easily obtained as

$$R(s) = \frac{N(s)}{D(s)} = \frac{\prod_{p=1}^{P-1}(s + z_{p,n})}{\prod_{p=1}^{P}(s + z_{p,d})} = \sum_{p=1}^{P} \frac{\alpha_p}{s + z_{p,d}}. \tag{4}$$

The poles $z_d = \{-z_{1,d}, ..., -z_{P,d}\}$ can be calculated directly as the zeros of the minimal state-space realization of $D(s)$, so the calculation of the α values reduces to a linear problem. In order to relocate the poles to a better position, a Sanathanan-Koerner (SK) iteration [3] can be applied, using an implicit weighting scheme. This means that the coefficients $d^{(t)}$ and $\tilde{d}^{(t)}$ of the weighted numerator $(N^{(t)}/D^{(t-1)})$ and denominator $(D^{(t)}/D^{(t-1)})$ are estimated, rather than the coefficients of the numerator $(N^{(t)})$ and denominator $(D^{(t)})$ themselves

$$\begin{aligned} &\arg \min_{d^{(t)}, \tilde{d}^{(t)}} \left(\sum_{k=0}^{K} \left| \frac{N^{(t)}(s_k)}{D^{(t-1)}(s_k)} - \frac{D^{(t)}(s_k)}{D^{(t-1)}(s_k)} H(s_k) \right|^2 \right) \\ &= \arg \min_{d^{(t)}, \tilde{d}^{(t)}} \left(\sum_{k=0}^{K} \left| \sum_{p=1}^{P} \frac{d_p^{(t)}}{s_k + z_{p,d}^{(t-1)}} - \left(\tilde{d}_0^{(t)} + \sum_{p=1}^{P} \frac{\tilde{d}_p^{(t)}}{s_k + z_{p,d}^{(t-1)}} \right) H(s_k) \right|^2 \right). \end{aligned} \tag{5}$$

This fact does not pose a problem, as the introduction of this weighting does not influence the zeros of $D^{(t)}$. The implicit scheme, however, is numerically more reliable, especially if the poles are not optimally chosen. The reader is referred to [2, 4] for more details about this procedure.

Experience with the original VF algorithm has shown that its convergence properties become severely impaired if the response to be fitted is contaminated with noise. It was shown in [5] that this problem is related to the adopted LS normalization where $\tilde{d}_0^{(t)}$ is set equal to 1. In [5], a modification to the (O)VF algorithm was introduced that alleviates these difficulties by improving the normalization of the transfer function coefficients and the linearization of the SK-iteration at the same time. As the iteration converges, it is assumed that $D^{(t-1)}(s_k)$ will approach $D^{(t)}(s_k)$, so an unbiased fitting would be achieved if $D^{(t)}(s_k)/D^{(t-1)}(s_k)$ approaches unity at all frequencies. In order to obtain this goal, a more relaxed non-triviality condition is added as an additional row in the system matrix

$$\Re e \left\{ \sum_{k=0}^{K} \left(\tilde{d}_0^{(t)} + \sum_{p=1}^{P} \frac{\tilde{d}_p^{(t)}}{s_k + z_{p,d}^{(t-1)}} \right) \right\} = K + 1. \tag{6}$$

This equation is given a LS weighting in relation to the size of H

$$weight = \|H(s)\| / (K + 1). \tag{7}$$

It was shown that this approach significantly improves the relocation of poles if the system equations are overdetermined, or when the data is corrupted with noise.

2. Relaxed SK Weighting

In this section an alternative way of relaxation, which is based on an explicit weighting of the SK-iteration, is proposed. Recall that the use of an explicit weighting corresponds to

solving the coefficients $c_p^{(t)}$ and $\tilde{c}_p^{(t)}$ of $N(s)$ and $D(s)$, provided that each equation is given a weighting as follows

$$\begin{aligned} &\arg \min_{c^{(t)},\tilde{c}^{(t)}} \left(\sum_{k=0}^{K} \left| \frac{N^{(t)}(s_k)}{D^{(t-1)}(s_k)} - \frac{D^{(t)}(s_k)}{D^{(t-1)}(s_k)} H(s_k) \right|^2 \right) \\ =&\arg \min_{c^{(t)},\tilde{c}^{(t)}} \left(\sum_{k=0}^{K} \left| w_k \left[\sum_{p=1}^{P} \frac{c_p^{(t)}}{s_k + a_p} - \left(\tilde{c}_0^{(t)} + \sum_{p=1}^{P} \frac{\tilde{c}_p^{(t)}}{s_k + a_p} \right) H(s_k) \right] \right|^2 \right). \end{aligned} \tag{8}$$

where $w_k = 1/D^{(t-1)}(s_k)$. In [6], 't Mannetje proposed to relax this weighting in each iteration by raising it to the power r, such that w_k in (8) is generalized as follows

$$w_k = \left(\frac{1}{D^{(t-1)}(s_k)} \right)^r = \left(\frac{\prod_{p=1}^{P}(s_k + a_p)}{\prod_{p=1}^{P}(s_k + z_{p,d}^{(t-1)})} \right)^r. \tag{9}$$

Clearly, $-z_{p,d}^{(t-1)}$ represent the zeros of $D^{(t-1)}(s_k)$, which are also equivalent to the poles of $D^{(t)}(s_k)/D^{(t-1)}(s_k)$. Note that (8) reduces to Levi's estimator [7] if $r = 0$, and to the classical SK-iteration if $r = 1$. In each iteration of the algorithm, the optimal choice of r can be determined by using standard optimization techniques. According to our practical experiments, the optimal value of r is usually located in the interval $0 < r < 2$. It was shown in [6] that this approach often improves the convergence properties of the explicitly-weighted SK iteration, particularly when the data is contaminated with noise.

3. Examples

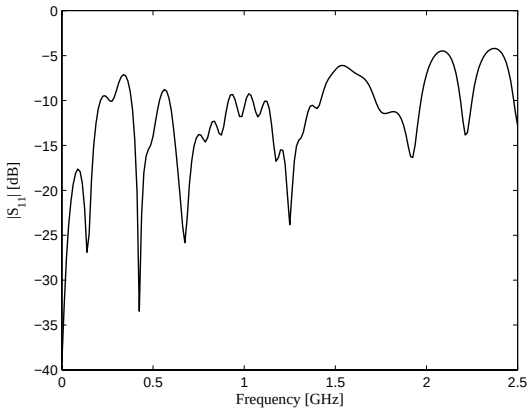
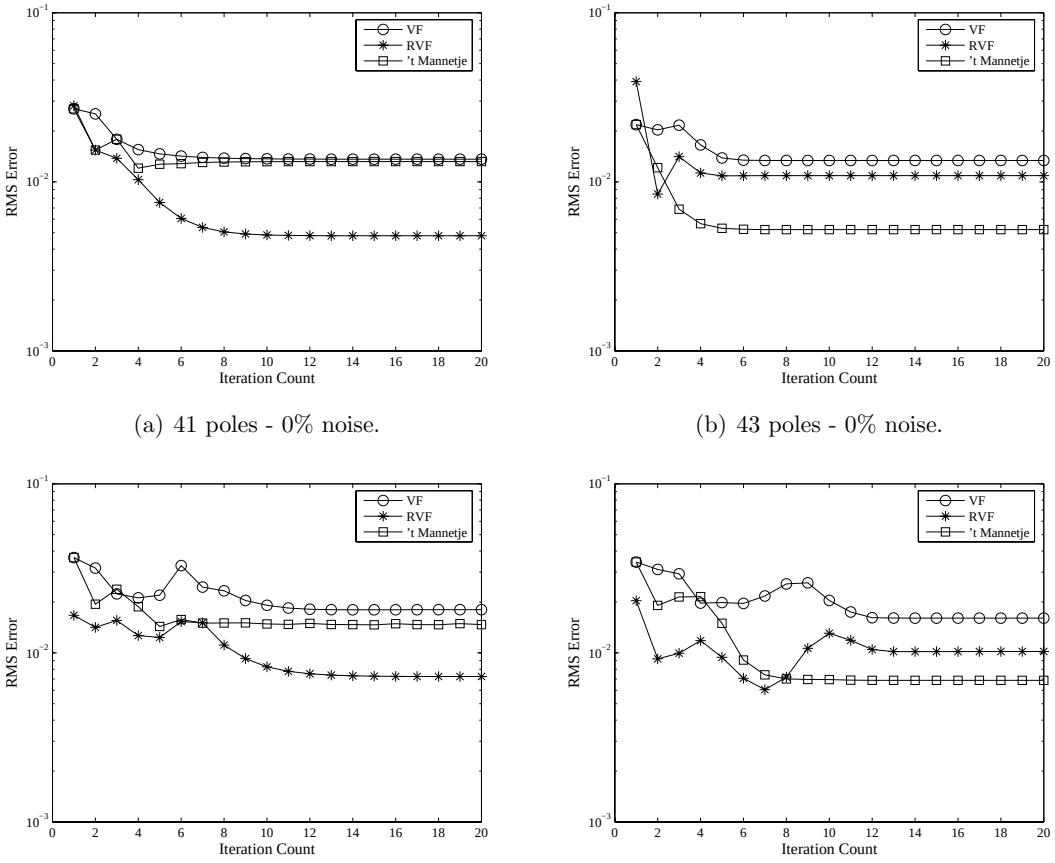


Figure 1. Magnitude of the data (S_{11}).

The reflection coefficient S_{11} of an RDRAM channel with 16 memory devices was simulated, and approximated from DC up to 2.5 GHz by a strictly proper transfer function. The data shows large reflection, as can be seen from the magnitude response in Figure 1. This frequency domain data is used to compare the VF and RVF approaches (using implicit weighting) to the method proposed by 't Mannetje (using explicit weighting).



(a) 41 poles - 0% noise.

(b) 43 poles - 0% noise.

(c) 41 poles - 1% noise.

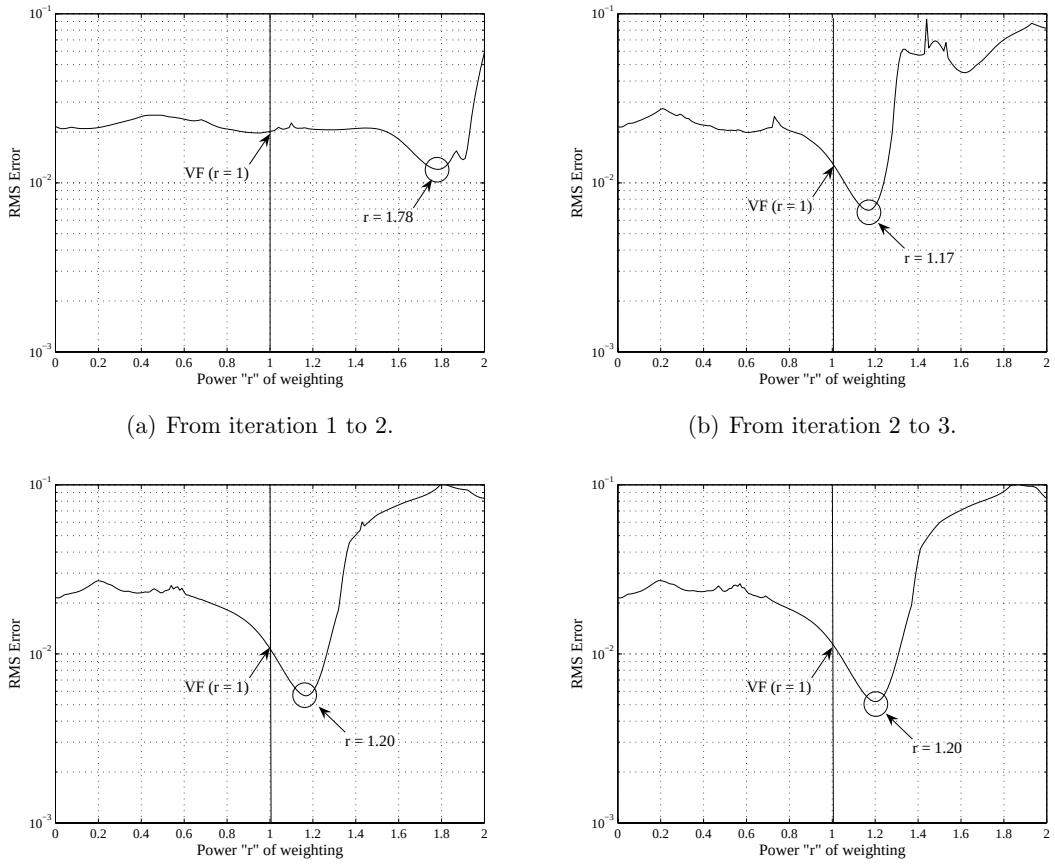
(d) 43 poles - 1% noise.

Figure 2. RMS error vs iteration count.

The starting poles are chosen as a prescribed set of complex conjugate pairs on a straight line with small real parts ($-0.01\omega_{\max}$), and their imaginary parts are equidistantly spread over the frequency range of interest. At DC, the complex conjugate pair of poles with zero-imaginary part is replaced by a single real pole in order to avoid a singular set of equations. In this example, all frequencies are scaled by 10^9. The methods proposed in this paper are used to relocate these poles, in order to minimize the fitting error. Unstable poles are flipped into the left-half plane during each iteration, in order to enforce stability of the poles.

Figure 2 shows the evolution of the RMS error in terms of iteration count:

- Figure 2(a): 41 starting poles, with no additional noise.
- Figure 2(b): 43 starting poles, with no additional noise.
- Figure 2(c): 41 starting poles, with 1% additional noise.
- Figure 2(d): 43 starting poles, with 1% additional noise.



(a) From iteration 1 to 2.

(b) From iteration 2 to 3.

(c) From iteration 3 to 4.

(d) From iteration 19 to 20.

Figure 3. Optimization of weighting for various iterations (43 poles - 0% noise).

In the general case, 't Mannetje gives a result lying somewhere in-between VF and RVF, as is shown in Figures 2(a) and (c). This appears to be the most representative situation for other examples and datasets as well. It is noted that in some cases, 't Mannetje can outperform VF and RVF (see Figures 2(b) and (d)), which indicates that the convergence of these methods is not guaranteed to be optimal. This also follows from the fact that RVF is less accurate when 43 poles are used (see Figure 2(b)), as compared to the situation when 41 poles are used (see Figure 2(a)). Even when additional noise is added to the data (see Figure 2(d)), 't Mannetje eventually leads to a better model than RVF. When considering iteration 3 of Figure 2(c) and iteration 4 of Figure 2(d), it can be seen that 't Mannetje is less accurate than VF. This interesting fact shows that the optimal value of r in a given iteration, does not necessarily imply that better results will be obtained in successive iterations. This indicates that a better result may occur if a sub-optimal choice is made at some point.

Figure 3 plots the RMS error of the fitting model against various choices of $0 < r < 2$ in different iterations. Specifically, the RMS error of the optimal choice of r (marked by an arrow and a circle) corresponds to the RMS errors shown in Figure 2(b). The behaviour of the curve is smooth, which indicates that the optimization problem is relatively easy. It

is also observed that the optimal error drops in successive iterations, and that it leads to better intermediate results when compared to the classical SK-iteration ($r = 1$).

It should be noted that the convergence of 't Mannetje can sometimes be impaired by ill-conditioning, which is caused by the explicit weighting. Such ill-conditioning may occur e.g. if the data is poorly observable, or when the initial poles are poorly specified. The reader is referred to [4] for more details about the numerical properties of implicit versus explicit weighting.

To our experience, it is very difficult to find a VF-based technique which works consistently better over the others. At this point, R(O)VF is considered to be the best approach, as it is numerically robust (partially due to the implicit weighting), and has the best convergence properties in many practical situations.

4. Conclusion

A variation of the Vector Fitting method, which is based on a relaxed explicit weighting of the system equations, is proposed ('t Mannetje). Its convergence properties are analyzed and compared to the classical VF and RVF formulations. The examples illustrate that neither technique works consistently better over the others. In the general case, RVF is the preferable approach because it is numerically more robust than 't Mannetje (because of the implicit weighting), and it often has better convergence properties than VF.

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